Minimization of the sum with the product constraint.
Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

For given $n \geq 3$ positive real $a_1, a_2, ..., a_n$ find $\mu := \min \sum_{i=1}^{n} x_i$,where $x_1, x_2, ..., x_n$ be any positive real numbers such that $x_1 x_2 ... x_n \geq \sum_{i=1}^{n} a_i x_i$.

Solution (without using partial derivatives and Lagrange multipliers).

1. First we will prove that $\mu = \min \sum_{i=1}^{n} x_i$, where positive $x_1, x_2, ..., x_n$ subject to claim $x_1 x_2 ... x_n = \sum_{i=1}^{n} a_i x_i$.Indeed, suppose that $\mu = x_1 + x_2 + ... + x_n$, where

$x_1 x_2 ... x_n > \sum_{i=1}^{n} a_i x_i \iff x_n (x_1 x_2 ... x_{n-1} - a_n) > \sum_{i=1}^{n-1} a_i x_i$.

Then $x_1 x_2 ... x_{n-1} - a_n > 0$ and, therefore, $x_n > \dfrac{\sum_{i=1}^{n-1} a_i x_i}{x_1 x_2 ... x_{n-1} - a_n}$.

Let $x'_n := \dfrac{\sum_{i=1}^{n-1} a_i x_i}{x_1 x_2 ... x_{n-1} - a_n}$ then $x_1 x_2 ... x_{n-1} x'_n = \sum_{i=1}^{n-1} a_i x_i + a_n x'_n$ and $x_n > x'_n \iff \mu \ = x_1 + x_2 + ... + x_{n-1} + x'_n$, that is contradiction.

2. Minimization.

Let $p_i, i = 1, 2, ..., n$ positive ideterminate parameters, such that $\sum_{i=1}^{n} a_i p_i = 1$ then by setting $x_i = p_i t_i, i = 1, 2, ..., n$ we obtain $x_1 x_2 ... x_n = \sum_{i=1}^{n} a_i x_i \iff$

$p_1 p_2 ... p_n t_1 t_2 .. t_n = \sum_{i=1}^{n} a_i p_i t_i$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} p_i t_i$.

Let $S := \sum_{i=1}^{n} p_i$ and $P := \prod_{i=1}^{n} p_i$.Since by Weighted AM-GM Inequality

(1) $\sum_{i=1}^{n} a_i p_i t_i \geq \prod_{i=1}^{n} t_i^{a_i p_i}$ and

(2) $\sum_{i=1}^{n} p_i t_i \geq S \prod_{i=1}^{n} t_i^{\frac{p_i}{S}}$

then we have $P \prod_{i=1}^{n} t_i = \sum_{i=1}^{n} a_i p_i t_i \geq \prod_{i=1}^{n} t_i^{a_i p_i} \iff \prod_{i=1}^{n} t_i^{1-a_i p_i} \geq \dfrac{1}{P}$ and from the other hand $\sum_{i=1}^{n} p_i t_i \geq S \prod_{i=1}^{n} t_i^{\frac{p_i}{S}}$.

Now we claim:

(3) $t_1 = t_2 = ... = t_n = t$ (Equality condition in **(1)** and **(2)**);

(4) $\dfrac{1 - a_1 p_1}{p_1} = \dfrac{1 - a_2 p_2}{p_2} = ... = \dfrac{1 - a_n p_n}{p_n}$.Let $k = \dfrac{1 - a_i p_i}{p_i}, i = 1, 2, ..., n$.

Then $p_i = \dfrac{1}{a_i + k}, i = 1, 2, ..., n,\ P = \prod_{i=1} \dfrac{1}{a_i + k}$and $S = \sum_{i=1}^{n} p_i \iff S =$

$\sum_{i=1}^{n}\dfrac{1}{a_i+k}$.

Since $\sum_{i=1}^{n} a_i p_i = 1$ then we obtain following equation for k :

(5) $\sum_{i=1}^{n}\dfrac{a_i}{a_i+k} = 1.$

Equation **(5)** always have unique positive solution which we denote k_*.

From now we assume that indeterminate parameters $p_1, p_2, ..., p_n$ are subject to the conditions $\sum_{i=1}^{n} a_i p_i = 1,\ p_1 p_2 ... p_n = \prod_{i=1}\dfrac{1}{a_i+k_*} \iff P = \prod_{i=1}\dfrac{1}{a_i+k_*}$,

and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n}\dfrac{1}{a_i+k_*} \iff S = \sum_{i=1}^{n}\dfrac{1}{a_i+k_*}$

Also, condition $P\prod_{i=1} t_i = \sum_{i=1}^{n} a_i p_i t_i$ gives us $Pt^n = \sum_{i=1}^{n} a_i p_i t \iff t = \dfrac{1}{\sqrt[n-1]{P}}$ and since $P = \prod_{i=1}\dfrac{1}{a_i+k_*}$ then $t = t_* := \sqrt[n-1]{\prod_{i=1}(a_i+k_*)}$.

Since $\prod_{i=1} t_i^{1-a_i p_i} \geq \dfrac{1}{P}$ then $\sum_{i=1}^{n} p_i t_i \geq S\prod_{i=1} t_i^{\frac{p_i}{S}} = S\left(\prod_{i=1} t_i^{k p_i}\right)^{\frac{1}{kS}} = S\left(\prod_{i=1} t_i^{1-a_i p_i}\right)^{\frac{1}{kS}} \geq S\left(\dfrac{1}{P}\right)^{\frac{1}{kS}}$.

Noting that $S = \sum_{i=1}^{n}\dfrac{1}{a_i+k_*} = \dfrac{1}{k}\sum_{i=1}^{n}\left(\dfrac{k_*+a_i}{a_i+k_*} - \dfrac{a_i}{a_i+k_*}\right) = \dfrac{n-1}{k_*}$, we finally obtain

$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} p_i t_i \geq S\left(\dfrac{1}{P}\right)^{\frac{1}{kS}} = S\left(\dfrac{1}{P}\right)^{\frac{1}{n-1}} \iff \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n}\dfrac{1}{a_i+k_*}\cdot \sqrt[n-1]{\prod_{i=1}(a_i+k_*)} \iff$

(6) $\sum_{i=1}^{n} x_i \geq \dfrac{n-1}{k_*}\sqrt[n-1]{\prod_{i=1}(a_i+k_*)}.$

Let $x_i^* := \dfrac{t_*}{a_i+k_*}, i = 1, 2, ..., n$.Then $\sum_{i=1}^{n} x_i^* = \dfrac{n-1}{k_*}\sqrt[n-1]{\prod_{i=1}(a_i+k_*)}$ and, therefore, $\mu = \dfrac{n-1}{k_*}\sqrt[n-1]{\prod_{i=1}(a_i+k_*)}$.

Remark. (About equation(5))

Let $Q(x) := \prod_{i=1}(x+a_i)$ then $\sum_{i=1}^{n}\dfrac{1}{x+a_i} = \dfrac{Q'(x)}{Q(x)}$ and equation $\sum_{i=1}^{n}\dfrac{a_i}{x+a_i} = 1 \iff \sum_{i=1}^{n}\left(1 - \dfrac{x}{x+a_i}\right) = 1 \iff x\sum_{i=1}^{n}\dfrac{1}{x+a_i} = n-1$ becomes $xQ'(x) = (n-1)Q(x)$.

Let $q_i := \sum_{1\leq k_1<...<k_i\leq i} a_{k_1}...a_{k_i}, i = 1, 2, ..., n$ then $Q(x) = x^n + q_1 x^{n-1} + ... + q_{n-1}x + q_n$ and $xQ'(x) = (n-1)Q(x) \iff nx^n + \sum_{i=1}^{n-1}(n-i)q_i x^{n-i} =$

$(n-1)\,x^n + \sum\limits_{i=1}^{n}(n-1)\,q_i x^{n-i} \iff$

(7) $x^n - q_2x^{n-2} - 2q_3x^{n-3} - ... - (n-2)\,q_{n-1}x - (n-1)\,q_n = 0.$

Thus, $\sum\limits_{i=1}^{n}\dfrac{a_i}{a_i+k} = 1 \iff k^n - q_2k^{n-2} - 2q_3k^{n-3} - ... - (n-2)\,q_{n-1}k - (n-1)\,q_n = 0.$

In particular, for $n=3$ we have $x^3 - q_2x - 2q_3 = 0.$

